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Calculus (Tutorial # 3)

Sequences and series

1. Let $\{a_n\} \subseteq \mathbb{R}$ be a sequence. Then there exists a monotone subsequence of $\{a_n\}$.

2. Let $\{a_n\}$ be a sequence such that $a_1 = 1$, $a_2 = 2$ and $a_n := \dfrac{a_{n-1} + a_{n-2}}{2}$ for $n \geq 3$. Then prove that $\{a_n\}$ is convergent by completing the following steps.

   (a) Using the monotone convergence theorem, show that $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are convergent subsequences of $\{a_n\}$.

   (b) Show that $\lim\limits_{n\to\infty}(a_{2n+1} - a_{2n}) = 0$.

   Also find $\lim\limits_{n\to\infty} a_n$.

3. Let $a_1 > 0$ is fixed and define the sequence $\{a_n\}$ by $a_{n+1} := \dfrac{1}{2}\Big(1 + \dfrac{1}{a_n}\Big)$ for $n \in \mathbb{N}$, then show that $\{a_n\}$ converges. Also find its limit.

4. Let $\{a_n\}$ be a bounded sequence of real numbers. Let

$$\begin{aligned} S &:= \{a : \ \exists \text{ a subsequence } \{a_{n_k}\} \text{ of } \{a_n\} \text{ such that } \lim_{k\to\infty} a_{n_k} = a\} \\ &:= \text{set of all subsequential limits of } \{a_n\}. \end{aligned}$$

   Then $\limsup a_n \in S$, $\liminf a_n \in S$ and $S \subseteq [\liminf a_n, \limsup a_n]$.

5. Show that the sequence $\{a_n\}$ defined by $a_n := \left(1 + \dfrac{1}{n}\right)^n$ for $n \in \mathbb{N}$ is convergent.

6. Let $\{a_n\} \subseteq \mathbb{R}$ be such that $a_{n+1} \leq a_n + \dfrac{1}{n}$ for all $n \geq 1$. Is it always necessary that $\{a_n\}$ has to converge? Justify your answer.

7. Find the limits of the following sequences $\{a_n\}$ for $n \in \mathbb{N}$.

   (a) $a_n = r^n$, for fixed r such that $|r| \leq 1$ and $r \neq -1$

   (b) $a_n = nr^n$, for fixed r such that $|r| < 1$

   (c) $a_n = r^{\frac{1}{n}}$, for fixed $r > 0$.

   (d) $a_n = n^{\frac{1}{n}}$

   (e) $a_n = \frac{a^n}{n!}$, for some fixed $a \in \mathbb{R}$

   (f) $a_n = \left(n^{\frac{1}{n}} - 1\right)^n$.

8. Can you find a sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = 0$ but $\{a_n\}$ is not a Cauchy sequence?

9. Prove: a sequence $\{a_n\} \subseteq \mathbb{R}$ has no convergent subsequence if and only if $\lim_{n\to\infty} |a_n| = \infty$.

10. For any bounded sequence $\{a_n\}$ of non-zero real numbers, prove that

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

Hence, show that convergence of $\left\{\frac{|a_{n+1}|}{|a_n|}\right\}$ implies the convergence of $\{|a_n|^{1/n}\}$.

11. Let $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$. Show that $\exists\ c_n > 0$, for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} c_n = \infty$ and $\sum_{n=1}^{\infty} c_n a_n < \infty$.

12. Let $a_n > 0$ such that $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$. What can you say about the convergence of the sequence $\{na_n\}$? What would be your conclusion if the second part of the hypothesis is removed i.e. $\sum_{n=1}^{\infty} a_n$ is divergent.

13. Can you find a sequence of positive real numbers $\{a_n\}$ such that $a_n \to 0$ but $\sum_{n=1}^{\infty} a_n^p$ is divergent for all $p \ge 1$?

14. Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is also convergent. Does the converse hold?

15. Test the convergence of the following series:

$(a)\ \sum_{k=0}^{\infty} \frac{e^{-k}}{\sqrt{k+1}}$ $(b)\ \sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ $(c)\ \sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1}$ $(d)\ \sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$(e)\ \sum_{n=1}^{\infty} \frac{1}{(n+2)\log(n+2)}$ $(f)\ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ $(g)\ \sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ $(h)\ \sum \frac{\log n}{\sqrt{n(n+1)}}$

$(i)\ \sum_{n=1}^{\infty} \frac{1}{2^n - n}$ $(j)\ \sum_{n=1}^{\infty} \log(1 + \frac{1}{n})$ $(k)\ \sum_{n=1}^{\infty} \frac{n^p}{e^n}$, for $p \ge 0$ $(l)\ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$(m)\ \sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ $(n)\ \sum_{n=1}^{\infty} \frac{1}{n \log n (\log\log n)^2}$ $(o)\ \sum_{n=1}^{\infty} \frac{2 + (-1)^n}{2^n}$ $(p)\ \sum_{n=1}^{\infty} ne^{-n^2}$

$(q)\ \sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{(n+1)^3 - 1}$ $(r)\ \sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ $(s)\ \sum_{n=1}^{\infty} \frac{1}{(\log n)^{1/n}}$ $(t)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $(u)\ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$

$(v)\ \sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n}$ $(w)\ \sum_{n=1}^{\infty} \left(1 - n\sin(\frac{1}{n})\right)$ $(x)\ \sum_{n=1}^{\infty} \frac{1}{n(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})}$

$(y)\ \sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ square} \\ \frac{1}{n^2} & \text{otherwise} \end{cases}$ $(z)\ \sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} \frac{1}{n^2}, & \text{if } n \text{ is odd} \\ -\frac{1}{n} & \text{if } n \text{ is even} \end{cases}$